Stability of Laminar Pipe Flows of Drag Reducing Polymer Solutions in the Presence of High-Phase-Velocity Disturbances

A theoretical study of high-phase-velocity disturbances in the laminar pipe flow of drag-reducing polymer solutions is reported. These disturbances are found to be more stable in a polymer solution with a sufficiently small relaxation time than in a Newtonian fluid. The opposite effect is predicted when the relaxation time exceeds a critical value, however. These results are shown to be consistent with a hydrodynamic instability explanation of the early turbulence phenomenon.

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SCOPE

The stability of laminar flows of drag-reducing polymer solutions has been considered by numerous investigators in recent years. Their work has been motivated in part by a realization that a close relationship may exist between laminar flow stability and turbulent flow drag reduction. In addition, Denn and Roisman (1969) have suggested that stability analyses may be useful in determining the rheological properties of the very dilute polymer solutions of greatest interest in applications of the drag reduction phenomenon. Finally, such analyses may provide explanations for the unusual behavior of dilute polymer solutions in transition flows (Paterson and Abernathy, 1972) and the early turbulence phenomenon (Forame et al., 1972), a deviation from Newtonian behavior which can be observed above a well-defined, subcritical Reynolds number in pipe flows of such solutions. Previous work has been concerned primarily with flows between rotating, concentric cylinders and between parallel, flat plates. Much less attention has been devoted to the stability of pipe flows of polymer solutions. This geometry is of particular interest, nevertheless, in that most early turbulence and drag reduction experiments have been conducted in pipes.

In this work a theoretical stability analysis of the laminar pipe flow of polymer solutions is given. Axisymmetric and rotationally symmetric disturbances are considered, as well as those which are independent of the direction of the undisturbed flow. The analysis is restricted to disturbances of large phase velocities. As in the case of pipe flows of Newtonian fluids, rigorous studies of this category of perturbation can be conducted with a minimum of mathematical complexity while serving to establish the mathematical nature of the stability problem (that is, whether or not a discrete, infinite set of eigenvalues exists). More importantly, the analysis shows how a polymer additive affects the stability of the disturbances considered over a broad range of viscoelastic parameters, whereas previous studies have in many cases been applicable to a limited range of these parameters.

CONCLUSIONS AND SIGNIFICANCE

All high-phase-velocity disturbances decay with time in the laminar pipe flow of a polymer solution, as is the case with laminar flows of Newtonian fluids in pipes and between parallel plates (Schensted, 1960). The polymer additive exerts a stabilizing influence on the disturbances considered (increases their rate of decay with time) when the solution relaxation time is very small, that is, when the additive concentration is very low. As the relaxation time is increased, however, a critical value is reached above which the polymer has a destabilizing effect. An example of this behavior is given in Figure 1 for a constant value of the product of solution relaxation time and wave number. The ordinate is the imaginary part of the normalized phase velocity, which becomes a larger negative number as the rate of decay of a disturbance increases. The abscissa is a ratio of solution relaxation time to Reynolds number. Newtonian behavior [predicted from Equation (31)] is indicated by the dotted line, while each of the other curves is for a polymer solution with particular rheological characteristics. Figure 1 also describes the behavior of high-phase-velocity disturbances in a hypothetical plug flow of the polymer solution, but this hypothetical flow does not appear to correspond to any physically significant case.

These results are significant in two respects. First, some previous investigators (Mook, 1972) have predicted that a polymer additive will stabilize a flow, while others (Platten and Schechter, 1970; Chan Man Fong and Walters, 1965) have concluded that the opposite effect may be observed. The present work suggests that such contradictory results may be attributable in part to consideration of only a limited range of viscoelastic parameters in some studies. Secondly, a deviation from Newtonian behavior has been observed in laminar pipe flows of drag reducing polymer solutions by a number of investigators (Forame, et al., 1972). The suggestion has been made that this so-called "early turbulence" phenomenon is due to a hydrodynamic instability. The present work substantiates this view. Changes in solution viscosity and polymer concentration which increase the predicted destabilization of high phase velocity disturbances are found by experiment to lower the Reynolds number for the onset of early turbulence.

THEORETICAL ANALYSIS

General Considerations

Two constitutive equations are employed in the present study to characterize polymer solutions. The first is a three-constant model, which is written

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\tau'^{ij} = \left(1+\frac{\lambda}{K}\frac{\partial}{\partial t}\right)(v^i{}_{,m}g^{mj}+v^j{}_{,m}g^{im}) \tag{1}$$

where

$$\tau'^{ij} = pg^{ij} + \tau^{ij}$$

All quantities are normalized by dividing by the velocity U of the undisturbed flow at the pipe centerline, the tube radius R, the quotient R/U, or $\mu U/R$, where μ denotes solvent viscosity. This equation has been used in previous studies of the stability of laminar boundary layers (Betchov, 1965). The second is a two-constant model, which may be thought of as a special case of Equation (1) obtained by setting $K = \infty$; that is,

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\tau'^{ij} = (v^i{}_{,m}g^{mj}+v^j{}_{,m}g^{im}) \tag{2}$$

This is sometimes called the Maxwell model (Betchov and Criminale, 1967). Equation (1) differs from the Oldroyd three-constant equation (Oldroyd, 1950) which is a quantitatively correct rheological description of some dilute polymer solutions when $K > 1$ (Darby, 1970) in that time derivatives $(\partial/\partial t)$ replace convected derivatives $(\delta/\delta t)$. The two are related by the equation

$$\frac{\delta\tau'^{ij}}{\delta t} = \frac{\partial\tau'^{ij}}{\partial t} + \tau'^{ij}{}_{,m}v^m - v^i{}_{,m}\tau'^{mj} - v^j{}_{,m}\tau'^{im} \tag{3}$$

Equation (2) is similarly related to the convected Maxwell model (White and Metzner, 1963) which has also been used with some success in studying flows of polymer solutions (Metzner, et al., 1966). Thus, Equations (1) and (2) are good approximations for the Oldroyd three-constant and convected Maxwell models, respectively, when $\partial/\partial t$ is the dominant term in $\delta/\delta t$. This condition is realized in a stability analysis of high-phase-velocity disturbances in pipe flows. The perturbation q to any steady state quantity Q can be expressed as terms of the form

$$q = \text{Real}\{\hat{q}(r)\exp[i\alpha(x-ct)+im\theta]\} \tag{4}$$

Here $\hat{q}$ denotes the complex magnitude of q. (All quantities are normalized as previously indicated.) Substitution of (4) into (3) shows the dominance of the time derivative for sufficiently large $|c|$, more specifically for $|c| >> 1$, $|c| >> |\alpha|$, and $|\alpha c| >> m$. This point has been further verified by deriving the governing differential equations for axisymmetric disturbances to laminar pipe flows of a convected Maxwell fluid, substituting into these equations the solution to this problem obtained below by using (2) and finding that the magnitudes of the additional terms introduced by having used the convected rather than time derivatives in the constitutive equation are negligibly small for sufficiently large $|c|$. The same result was not found to be valid for small $|c|$ in general. Earlier investigations (Derman, 1967; Betchov, 1965) in which time derivatives have been used in place of convected derivatives for a wide range of $|c|$ may not be applicable to polymer solutions, therefore.

Following most previous investigators of the pipe flow stability of Newtonian fluids (Pekeris, 1948; Corcos and Sellars, 1959; Salwen and Grosch, 1972) the so-called "timewise" problem for infinitesimal disturbances is treated. The disturbance is assumed to be of the form of (4), where α and m are real while c is complex in general. The magnitudes of the fluctuating quantities (pressure, stresses, velocities) are assumed small compared to the steady state values, and products of fluctuating quantities are taken to be negligible. When combined with the continuity and momentum equations and the constitutive Equation (1) these assumptions give the following governing differential equations for the complex magnitudes of the perturbation quantities:

$$\hat{u}' + \frac{\hat{u}}{r} + \frac{im\hat{v}}{r} + i\alpha\hat{w} = 0; \tag{5}$$

$$i\alpha(1-r^2-c)\hat{u} = -\hat{p}' + \frac{1}{N_c}\left\{\hat{u}'' + \frac{\hat{u}'}{r} - \left(\alpha^2+\frac{m^2+1}{r^2}\right)\hat{u} - \frac{2im}{r^2}\hat{v}\right\}; \tag{6}$$

$$i\alpha(1-r^2-c)\hat{v} = \frac{-im}{r}\hat{p} + \frac{1}{N_c}\left\{\hat{v}'' + \frac{\hat{v}'}{r} - \left(\alpha^2+\frac{m^2+1}{r^2}\right)\hat{v} + \frac{2im}{r^2}\hat{u}\right\}; \tag{7}$$

$$i\alpha(1-r^2-c)\hat{w} - 2r\hat{u} = -i\alpha\hat{p} + \frac{1}{N_c}\left\{\hat{w}'' + \frac{\hat{w}'}{r} - \left(\alpha^2+\frac{m^2}{r^2}\right)\hat{w}\right\}; \tag{8}$$

where

$$N_c = \frac{N(1-i\alpha c\lambda)}{\left(1-i\dfrac{\alpha c\lambda}{K}\right)} \tag{9}$$

Here N denotes Reynolds number and $\hat{p}$, $\hat{u}$, $\hat{v}$, and $\hat{w}$ the (normalized) complex magnitudes of the disturbance pressure and of the radial, azimuthal, and axial velocity components, respectively. The corresponding results for the Maxwell fluid are obtained by setting $K = \infty$ in Equation (9). Equations (5) to (8) differ from those for a Newtonian fluid only in that N_c takes the place of N.

Axisymmetric Disturbances

The disturbances to pipe flows of Newtonian fluids which have been most extensively studied are those for which $\hat{v} = 0$. The eigenvalue problem associated with disturbances of this type may be formulated in terms of a perturbation stream function with complex magnitude

$\hat{\phi}$, defined as follows:

$$\hat{u} = -i\alpha\hat{\phi}\,;\ \hat{w} = \frac{1}{r}\frac{d}{dr}(r\hat{\phi}). \qquad (10)$$

$\hat{\phi}$ satisfies the continuity condition (5). Substitution of $\hat{\phi}$ into (6) and (8) and combination of the resulting equations to eliminate $\hat{p}$ gives

$$i\alpha N_c\,(1 - r^2 - c)y = (L_1 - \alpha^2)y, \qquad (11)$$

where

$$y = (L_1 - \alpha^2)\hat{\phi} \qquad (12)$$

and

$$L_m = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} \qquad (13)$$

The boundary conditions are

$$\hat{\phi}'(0) = \text{finite};\quad \hat{\phi}(0) = \hat{\phi}(1) = \hat{\phi}'(1) = 0 \qquad (14)$$

The solution for $\hat{\phi}$ may be expressed as a linear combination of an inviscid part for which $y = 0$ and a viscous part for which y is not trivial. The former is

$$\hat{\phi}_i = C_1 J_1(i\alpha r) \qquad (15)$$

in light of the fact that $\hat{\phi}$ is regular at the origin. The latter is related to y by the following equation, which follows from application of the method of variation of parameters (Ford, 1955) to Equation (12):

$$\hat{\phi}_v = \frac{\pi}{2}\int_0^r \{t[J_1(i\alpha t)Y_1(i\alpha r) - Y_1(i\alpha t)J_1(i\alpha r)]\}\,y dt \qquad (16)$$

y is obtained from (11) by making the following change of variables:

$$F = ry;\quad z = \sqrt{-i\alpha N_c}\,r^2;\quad k = -\frac{\alpha^2 + i\alpha N_c(1-c)}{4\sqrt{-i\alpha N_c}} \qquad (17)$$

The resulting differential equation is

$$\frac{d^2F}{dz^2} + \left(\frac{k}{z} - \frac{1}{4}\right)F = 0 \qquad (18)$$

The solution which is regular at the origin is the Whittaker function $M_{k,1/2}(z)$ (Slater, 1960), which may be expressed as an infinite sum of Bessel functions:

$$M_{k,1/2}(z) = 4\left\{\frac{2\sqrt{kz}\,J_1(2\sqrt{kz})}{4k} + \frac{z^2}{24k}J_2(2\sqrt{kz}) + \ldots\right\} \qquad (19)$$

The leading term is large compared to all higher order terms when

$$\left|\frac{1}{6}\sqrt{-i\alpha N_c}\sqrt{\frac{i\,N_c}{\alpha + i\,(1-c)\,N_c}}\right| << 1 \qquad (20)$$

[The validity of approximating $M_{k,1/2}(z)$ by the first term in the series in (19), when a condition analogous to (20) is satisfied, has been confirmed rigorously (Romanovskaya and Khalfina, 1967).] Condition (20) is always satisfied for an Oldroyd fluid when the magnitude of the phase velocity is sufficiently large. More precisely $|c|$ must be large enough, for a given wave number disturbance in a flow of given Reynolds number, that the quotient $\frac{\alpha NK}{|c|}$ is very small compared to unity. An approximate solution for $\hat{\phi}$ in these circumstances is obtained by putting the first term of (19) into (16), giving

$$(\alpha^2 + \beta^2)\left(\frac{i\alpha k}{4\beta\pi}\right)\hat{\phi}_v = \beta^2 J_1(i\alpha r) + \frac{\alpha}{i\pi}J_1(\beta r) \qquad (21)$$

where

$$\beta^2 = -\alpha^2 - i\alpha N_c(1-c) \qquad (22)$$

Thus, an approximate solution for $\hat{\phi}$ at large $|c|$ is

$$\hat{\phi} = C_2 J_1(i\alpha r) + C_3 J_1(\beta r) \qquad (23)$$

The boundary conditions on $\hat{\phi}$ and $\hat{\phi}'$ at $r = 0$ and on $\hat{\phi}$ at $r = 1$ are satisfied by taking

$$C_2 = 1$$
$$C_3 = -\frac{J_1(i\alpha)}{J_1(\beta)} \qquad (24)$$

The condition on $\hat{\phi}'$ at $r = 1$ imposes the additional requirement that

$$\begin{vmatrix} J_1(i\alpha) & J_1(\beta) \\ J_1'(i\alpha) & J_1'(\beta) \end{vmatrix} = 0, \qquad (25)$$

or more simply

$$\frac{\beta J_0(\beta)}{J_1(\beta)} = \frac{i\alpha J_0(i\alpha)}{J_1(i\alpha)} \qquad (26)$$

Introducing the asymptotic expression for a Bessel function of large argument (Abramowitz and Stegun, 1965) on the left-hand side of (26) and the relationship between $J_n(i\alpha)$ and $I_n(\alpha)$ on the right-hand side gives

$$\frac{\beta\cos\left(\beta - \frac{\pi}{4}\right)}{\cos\left(\beta - \frac{3\pi}{4}\right)} \sim \frac{\alpha I_0(\alpha)}{I_1(\alpha)} \qquad (27)$$

The quotient on the right-hand side of (27) approaches 2.0 as $\alpha \to 0$ and approaches α as α becomes large. Thus, (27) will be satisfied for $|\beta|$ large compared to unity and $|\alpha|$ (that is, large phase velocity and Reynolds number) when $\cos(\beta - \pi/4)$ is small compared to unity. Writing $\beta - \pi/4 = n\pi + \pi/2 - e$, where e is a small quantity, and substituting into (27) gives

$$e \sim \frac{\alpha I_0(\alpha)/I_1(\alpha)}{n\pi + \frac{3\pi}{4}} \qquad (28)$$

or

$$\beta \sim n\pi + \frac{3\pi}{4} - \frac{\alpha I_0(\alpha)/I_1(\alpha)}{n\pi + \frac{3\pi}{4}} = K_n \qquad (29)$$

Solving (29) for c and imposing the condition that the Oldroyd fluid behavior approaches that of a Newtonian fluid as $\lambda \to 0$, there results

$$c \sim -\frac{i}{2\alpha\lambda}\left[1+\frac{\lambda(K_n^2+\alpha^2)}{K\,N}\right]+\frac{1}{2}$$

$$+\frac{1}{2\alpha\lambda}\sqrt[4]{\left\{1+\left[\frac{\lambda(K_n^2+\alpha^2)}{K\,N}\right]^2-\alpha^2\lambda^2-2\left(2-\frac{1}{K}\right)\frac{\lambda(K_n^2+\alpha^2)}{N}\right\}^2+4\alpha^2\lambda^2\left[1-\frac{\lambda(K_n^2+\alpha^2)}{K\,N}\right]^2}$$

$$\times \exp\left\langle \frac{i\pi}{2}-\frac{i}{2}\tan^{-1}\left\{\frac{2\alpha\lambda\left[1-\dfrac{\lambda(K_n^2+\alpha^2)}{K\,N}\right]}{1+\left[\dfrac{\lambda(K_n^2+\alpha^2)}{K\,N}\right]^2-\alpha^2\lambda^2-2\left(2-\dfrac{1}{K}\right)\dfrac{\lambda(K_n^2+\alpha^2)}{N}}\right\}\right\rangle \quad (30)$$

The corresponding result for a Newtonian fluid is

$$c \sim 1-\frac{i(K_n^2+\alpha^2)}{\alpha\,N}=1-\frac{i\lambda(K_n^2+\alpha^2)}{N}\times\frac{1}{\alpha\lambda}. \quad (31)$$

A number of significant conclusions follow from (30) and (31). An infinite set of discrete eigenvalues are associated with the problem of axisymmetric disturbances in the pipe flow of an Oldroyd liquid. The high phase velocity disturbances decay with time. That is, the imaginary part of (30) is negative when $|c| >> 1$, which requires $\alpha\lambda << 1$. The variation of the imaginary part of c with $\frac{\lambda(K_n^2+\alpha^2)}{N}$ is shown in Figure 1 for $\alpha\lambda = 0.01$ and a range of values of $K > 1$. [Both experimental (Darby, 1970) and theoretical (Lumley, 1971) studies indicate that the effect of a polymer additive is to increase K from its Newtonian value of unity.] These plots are typical of results for all $\alpha\lambda << 1$. The polymer additive increases the rate of decay of high-phase-velocity disturbances for sufficiently small $\frac{\lambda(K_n^2+\alpha^2)}{N}$ and has the opposite effect

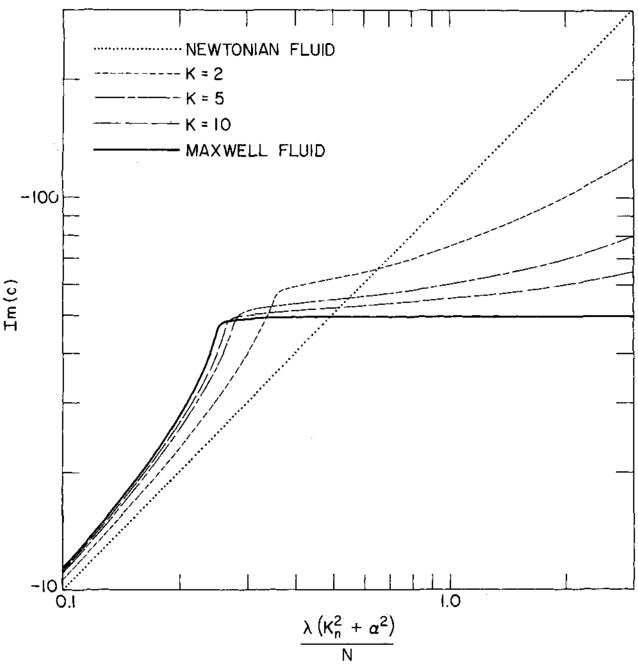

Fig. 1. Imaginary part of the (normalized) phase velocity of axisymmetric disturbances as a function of K and a ratio of (normalized) relaxation time to Reynolds number.

for large values of this parameter. This behavior can also be anticipated by expanding (30) for the limiting cases of small and large $\frac{\lambda(K_n^2+\alpha^2)}{N}$. For small values

$$c \sim 1+\frac{\lambda(K_n^2+\alpha^2)}{N}\left(1-\frac{1}{K}\right)-\frac{i(K_n^2+\alpha^2)}{\alpha N}-\frac{i\lambda(K_n^2+\alpha^2)^2}{\alpha N^2}\left(1-\frac{1}{K}\right); \quad (32)$$

and the negative imaginary part of c in this expansion exceeds that in (31). For very large $\frac{\lambda(K_n^2+\alpha^2)}{N}$, the negative imaginary part of c from (30) is asymptotic to $\frac{i(K_n^2+\alpha^2)}{\alpha NK}$, which will be less than that from (31) for the physically significant cases of $K > 1$. The critical condition which separates regions of stabilization and destabilization is obtained by equating negative imaginary parts of c from (30) and (31), giving

$$\frac{\lambda(K_n^2+\alpha^2)}{N}=\frac{T\,\mu(K_n^2+\alpha^2)}{R^2\,\rho}=f(K) \quad (33)$$

$f(K)$ is a function of K which is seen in Figure 1 to be of the order of 0.5. T denotes the dimensional relaxation time of the polymer solution and ρ its density.

A similar treatment of axisymmetric disturbances in a convected Maxwell fluid is valid when (20) is satisfied for $K = \infty$. That is to say, the quotient $\left|\frac{\alpha N}{c}\right|$ must be small compared to unity when $|\alpha c\lambda| \leq 1$; and $|\alpha^2\lambda N|$ must be small compared to unity for large values of $|\alpha c\lambda|$. These conditions may always be satisfied when the phase velocity is sufficiently large and the wave number sufficiently small. In these circumstances c and its asymptotic behavior at small and large $\frac{\lambda(K_n^2+\alpha^2)}{N}$ are obtained by setting $K = \infty$ in Equation (30). The relationship of the imaginary part of c to $\frac{\lambda(K_n^2+\alpha^2)}{N}$ for disturbances satisfying condition (20) in a convected Maxwell fluid is plotted in Figure 1.

It is of interest to note in retrospect that these predictions for axisymmetric disturbances in fully-developed pipe flow also describe the behavior of such disturbances in a hypothetical, fully-developed plug flow. That is to say, Equation (23) can be shown to represent the solution for $\hat{\phi}$ in a flow where the (normalized) unperturbed flow velocity in the axial direction is equal to unity. This correspondence in results for high phase velocity disturbances in the hypothetical case, and in the physically significant case of a parabolic profile analyzed above, was not anticipated a priori. Moreover, the hypothetical problem does not appear to correspond to any real flow situation involving a liquid. A fully-developed plug flow in a pipe could be realized only in a fluid of zero viscosity. A real liquid may approximate plug flow at the entrance to a pipe, but this flow is not fully-developed. It has a velocity gradient in the axial direction, as well as both axial and radial velocity components near the wall. Consequently, the stability equations for the entrance region will contain terms which are ignored in treating a hypothetical, fully-developed plug flow.

Rotationally Symmetric Disturbances

These are disturbances which are independent of the azimuthal coordinate, but for which $\hat{v}$ is not trivial. The governing differential equation for $\hat{v}$ obtained from (7) by setting $m = 0$, is identical to Equation (11) for y. The boundary conditions are

$$\hat{v}(0) = \text{finite}; \quad \hat{v}(1) = 0 \tag{34}$$

Making the change of variable

$$G = r\hat{v} \tag{35}$$

and introducing the definitions of k and z the following eigenvalue problem is obtained:

$$\frac{d^2G}{dz^2} + \left(\frac{k}{z} - \frac{1}{4}\right) G = 0 \tag{36}$$

$$G(0) = G(1) = 0 \tag{37}$$

From the analysis at the axisymmetric case it follows that

$$G \sim C_4\sqrt{kz}\, J_1\ (2\sqrt{kz}) \tag{38}$$

when the boundary condition at the origin is satisfied. The wall boundary condition necessitates that

$$J_1(2\sqrt{kz})\,|_{r=1} \sim 0 \tag{39}$$

or

$$4kz|_{r=1} \sim B_n^2 \tag{40}$$

where B_n is the nth zero of J_1. Solution of (40) for c gives the same result as in the axisymmetric case except that B_n replaces K_n [defined by (29)]. Thus, rotationally and axially symmetric disturbances of high phase velocity in Oldroyd and Maxwell fluids are similar in a number of respects. They decay at all Reynolds numbers, are more stable than in Newtonian fluids for small λ, and are less stable for sufficiently large λ.

Cross Disturbances

High phase velocity disturbances which are independent of x (cross disturbances) are analyzed by defining the complex magnitude of a perturbation stream function as follows:

$$\hat{v} = \frac{i}{m}\hat{\psi}'; \quad \hat{u} = \frac{\hat{\psi}}{r} \tag{41}$$

Substitution of these relationships into Equations (6) and (7), and elimination of $\hat{p}$ gives

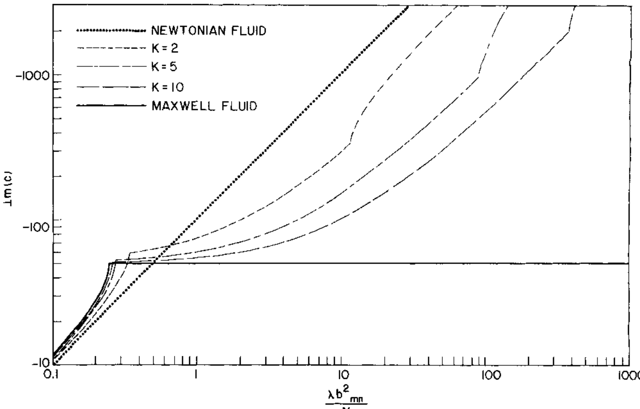

Fig. 2. Imaginary part of the (normalized) phase velocity of cross disturbances as a function of K and a ratio of (normalized) relaxation time to Reynolds number.

$$i\alpha c N_c(L_m\hat{\psi}) = -L_m(L_m\hat{\psi}) \tag{42}$$

The boundary conditions are

$$\hat{\psi}(0) = \hat{\psi}(1) = \hat{\psi}'(1) = 0; \quad \hat{\psi}'(0) = \text{finite} \tag{43}$$

The solution to (42) which satisfies the boundary conditions at $r = 0$ $(m \neq 0)$ is

$$\hat{\psi} = C_5 r^m + C_6 J_m(br) \tag{44}$$

where C_5 and C_6 are arbitrary constants and

$$b = \sqrt{i\alpha c N_c} \tag{45}$$

The boundary conditions at $r = 1$ are satisfied for

$$J_{m+1}(b) = 0 \tag{46}$$

Denoting by b_{mn} the nth eigenvalue of (46), which must be real (Watson, 1945), the following expression for the phase velocity is obtained:

$$c = \frac{i}{2\alpha\lambda}\left\{-1 - \frac{\lambda\, b_{mn}^2}{N\,K} \pm \sqrt{\left[1 + \frac{\lambda\, b_{mn}^2}{N\,K}\right]^2 - \frac{4\lambda\, b_{mn}^2}{N}}\right\} \tag{47}$$

Figure 2 displays this result for a range of values of K, as well as the corresponding results for Newtonian and convected Maxwell fluids, for $\alpha\lambda = 0.01$. As in the cases of axisymmetric and rotationally symmetric disturbances, the flow is stabilized by viscoelastic effects of small magnitude and destabilized when these effects become sufficiently large.

DISCUSSION

Derman (1967) has indicated that a polymer additive can destabilize pipe flow. The theoretical basis for this assertion is unsatisfactory in light of the present work. The assumption was made that Equation (2) validly describes a polymer solution for disturbances of all phase velocities, whereas the validity of this assumption can be demonstrated only for the large phase velocity cases treated above. Also, a two-term approximation to the solution of the stability problem was obtained by the Galerkin method; and convergence to the exact solution was not demonstrated by examining additional terms.

Analyses employing realistic constitutive equations and rigorous mathematical techniques have been conducted for the flow between rotating, concentric cylinders (Ginn and Denn, 1969). The polymer additive is predicted to exert a stabilizing or destabilizing influence, depending on the magnitudes of the relevant viscometric and geometric quantities. Substantial destabilization is anticipated when polymer concentration becomes large. These results resemble those of the present work for pipe flows. It would be surprising, therefore, if similar behavior were not observable in other geometries such as the flow between parallel plates. Some recent theoretical investigations (Mook, 1972) have indicated that only stabilization may occur in this geometry, however. In light of the present work and that of Ginn and Denn, such results may be attributable to consideration of only a limited range of viscoelastic and geometric parameters.

Two types of experiments reported recently indicate that a destabilizing influence on pipe flows can be exerted by a polymer additive. Paterson and Abernathy (1972) studied the laminar-to-turbulent transition in pipe flows of aqueous polyethylene oxide solutions. They found that, for low-disturbance pipe inlet conditions, the polymer additive decreased the transition Reynolds number in their system from the 8,000 to 11,000 range to as low as 2,900 to 3,300. They concluded from these results that the additive exerts a destabilizing influence on small disturbances in the flow.

Experiments conducted by the present author and others (Forame et al., 1972; Little and Wiegard, 1970; Ram and Tamir, 1964) have shown that the pipe flow of dilute polymer solutions at subcritical Reynolds numbers may deviate from Newtonian behavior above a well-defined onset wall shear stress. A continuous transition from this early turbulence regime to turbulent flow with reduced drag replaces the abrupt laminar-to-turbulent transition observed with Newtonian fluids. The suggestion has been made (Hansen et al., 1972) that early turbulence is a hydrodynamic instability phenomenon, and the theoretical results presented above are consistent with this view. Equation (30) predicts that, for $\frac{\lambda(K_n^2 + \alpha^2)}{N}$ above the critical value, the destabilizing effect of the polymer increases with K and with the quotient λ/N, which is equal to $T\mu/R^2\rho$. K is expected to increase (Lumley, 1971) and T to increase to a maximum and then remain constant or decrease, with increasing concentration (Kusamizu, et al., 1968; Hansen and Little, 1971; Meister and Biggs, 1969). T increases with solvent viscosity. Early turbulence experiments indicate that the degree of destabilization of the flow, as measured by the decrease of the onset Reynolds number from the critical Reynolds number for a Newtonian fluid, increases with increasing solvent viscosity, while it first increases and then decreases with increasing polymer concentration.

NOTATION

B_n = nth zero of J_1
b_{mn} = nth zero of J_{m+1}
c = phase velocity
e = defined by Equation (28)
F = defined by Equation (17)
$f(K)$ = the value of $\frac{\lambda(K_n^2 + \alpha^2)}{N}$ at which the imaginary part of the phase velocity in a Newtonian fluid is equal to that in a polymer solution characterized by a particular value of K
G = $r\hat{v}$
g^{ij} = conjugate metric tensor
k = defined by Equation (17)
K = ratio of relaxation to retardation times of a polymer solution
K_n = defined by Equation (29)
L_m = operator defined by Equation (13)
m, n = integers
N = Reynolds number
N_c = defined by Equation (9)
p = pressure; $\hat{p}$ = complex magnitude of perturbation pressure
R = (dimensional) tube radius
r = radial coordinate
T = (dimensional) relaxation time of the polymer solution
t = time
U = (dimensional) velocity of the undisturbed flow at the pipe centerline
$\hat{u}, \hat{v}, \hat{w}$ = complex magnitudes of the perturbation fluid velocities in the radial, azimuthal, and axial directions, respectively
$v^i{}_{,m}$ = covariant derivative of the ith velocity component
x = axial coordinate
y = defined by Equation (12)
z = defined by Equation (17)
$(\ )'$ = denotes differentiation with respect to r

Greek Letters

α = wave number
β = defined by Equation (22)
$\delta/\delta t$ = convected derivative
λ = relaxation time of the polymer solution
θ = azimuthal coordinate
$\hat{\phi}$ = complex magnitude of perturbation stream function for axisymmetric disturbances; $\hat{\phi}_i$ = inviscid part; $\hat{\phi}_v$ = viscous contribution
$\hat{\psi}$ = complex magnitude of perturbation stream function for cross disturbances
τ^{ij} = ijth component of stress tensor
μ = (dimensional) viscosity of polymer solution

All quantities are normalized by dividing by U, R, R/U, or $\mu U/R$ unless noted to be dimensional quantities.

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Liquid-Liquid Contacting in Unbaffled, Agitated Vessels

Drop size and dispersed-phase holdup were measured for liquid dispersions in covered, unbaffled agitated vessels with no gas-liquid interface and hence no vortex for batch and continuous flow. Vessel diameters of 0.245 and 0.372 m, turbine impellers of diameters 0.0762 and 0.127 m in two locations, four organic- and three water-dispersed systems of a wide range of properties, and a wide range of operating conditions were studied. The data for average drop diameter and dispersed-phase holdup, from which specific interfacial area may be computed, were successfully correlated through modification of the theories of maximum drop size in an isotropically turbulent fluid.

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SCOPE

Vessels for mechanical agitation of liquids in most processes are usually operated batchwise and open to the air with an air-liquid surface, commonly with a rotating impeller on an axially arranged shaft. Under these conditions, unless the vessel is suitably baffled, a vortex will form at all but the lowest agitator speeds. Rushton (1951) has shown that when a vortex forms, it is impossible to scale up the power requirement for agitation, even with geometrically similar vessels. Since the vortex can be eliminated by introducing four wall baffles (Mack and Kroll, 1948), almost all subsequent work with agitated vessels has been done with baffles installed.

In liquid-liquid extraction operations, two immiscible liquids are vigorously agitated so as to disperse one of them into tiny droplets of large interfacial area. Since one of the liquids is invariably organic, it is desirable to operate the agitated vessel covered; and since the operations usually require continuous flow of the liquids, it is easy to operate the vessel completely filled with liquid,

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